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HEAT TRANSFER WITH LAMINAR FLOW IN CONCENTRIC ANNULI WITH CONSTANT AND ARBITRARY VARIABLE AXIAL WALL TEMPERATURE

by

R. Viskanta

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HEAT TRANSFER WITH LAMINAR FLOW IN CONCENTRIC ANNULI WITH CONSTANT AND ARBITRARY VARIABLE AXIAL WALL TEMPERATURE

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HEAT TRANSFER WITH LAMINAR FLOW IN CONCENTRIC ANNULI WITH CONSTANT AND ARBITRARY VARIABLE AXIAL WALL TEMPERATURE

by

R. Viskanta

ABSTRACT

An analysis has been performed to determine the heat transfer characteristics for laminar forced-convection flow in a concentric annulus with prescribed surface temperatures. Three distinct problems were considered: (a) wall temperature prescribed at both the inside and outside wall; (b) inside wall temperature prescribed and the outside wall insulated; and (c) inside wall insulated and outside wall temperature prescribed. The solution for temperature distribution was similar to that obtained by Graetz for laminar heat convection in a pipe with uniform wall temperature. Expressions are presented for heat flux, mixing cup temperature, and Nusselt number as a function of downstream position. Eigenvalues and eigenfunctions were computed on an analogue computer for several values of the ratio of the inside to the outside radii for the above boundary conditions. Mixing cup temperatures, local and fully developed Nusselt numbers, and thermal entry lengths are presented graphically.

The solution of Problem (a) was extended to the situation in which the temperatures of the inside and outside walls of the annulus are not equal. By utilizing the method of superposition and the solutions already obtained for Problem (a), the temperature distributions were determined. By way of illustration, heat fluxes were calculated for several values of the temperature ratio $(T_{w_i} - T_0)/(T_{w_0} - T_0)$.

Results were then generalized to apply to the situation of arbitrary longitudinal variation of the wall temperatures of the annulus. As an illustration of the method, an extension is explicitly given for a linear increase of wall temperature with axial distance.

1. INTRODUCTION

The problem of laminar forced-convection heat transfer is of considerable practical interest and has been studied extensively since 1883. The heat transfer in a concentric annulus is a natural generalization of the Graetz problem, since flow between two parallel plates and in a pipe are special cases for values ∞ and 0, respectively, of the parameter $r_i/(r_0 - r_i)$. Most of the existing analyses for laminar flow and heat transfer in passages have been confined to circular tubes or parallel plates. These passages have been analyzed extensively because their simplicity makes them amenable to analysis. In recent years the problems associated with the use of odd-shaped coolant passages in heat exchangers, in heterogeneous nuclear reactors, and in other applications have made the process of heat transfer in an annulus of engineering importance.

It is assumed here that the fluid with constant physical properties enters the annulus with a uniform temperature and a fully developed laminar velocity profile, and up to some point ($x = 0$) the fluid is isothermal. Three distinct problems are considered:

(a) for $x > 0$ the wall temperatures are prescribed at both the inner and the outer walls;

(b) for $x > 0$ the inner wall temperature is prescribed and the outer wall is insulated; and

(c) for $x > 0$ the inner wall is insulated and a temperature is prescribed at the outer wall.

In Section 2 of this report, consideration is given to problems with constant prescribed wall temperatures. In addition, for Problem (a) the assumption is made that the wall temperatures are the same. In Section 3, Problem (a) of Section 2 is generalized, and solutions are obtained with different, but constant wall temperatures prescribed along each of the two walls. In Section 4 of this report the problems are generalized to the situation of an arbitrary axial variation of the surface temperatures.

To the author's knowledge, laminar flow heat transfer in an annulus with prescribed wall temperatures has been studied only by Murakawa.[1-4]* In these references, Murakawa has presented integral equation formulation as well as some experimental results for water heated from the inside wall with the outside wall of the annulus being insulated. However, Murakawa[3] has carried his solutions to the point of numerical calculation only for

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*A general and complete study on laminar flow heat transfer in an annulus by Lundberg _et al_.[20] came to the author's attention when this report was in press.

Problem (b) and for one value of the ratio of the inside to the outside radius of the annulus. A more extensive bibliography of similar problems for pipes and parallel plates can be found in Refs. 5-8.

The analysis which is made here is similar in mathematical approach to that presented by Graetz[9] for laminar forced convection in a round pipe with isothermal wall. The classical treatment of this problem by Graetz utilizes separation of variables which reduces the energy equation to a Sturm-Liouville equation. After the eigenfunctions and eigenvalues of the Bessel-type equation have been determined, the heat transfer parameters of interest can be readily determined. The first eigenfunction gives the solution far from the entrance of the annulus and an increasing number of eigenfunctions are required to obtain accurate temperature distribution as the distance from the entrance is decreased.

2. HEAT TRANSFER IN AN ANNULUS WITH PRESCRIBED CONSTANT WALL TEMPERATURE

2.1 Analysis

2.1.1 Mathematical Statement of the Problem

The coordinates and geometry of the system are shown in Figure 1. Fluid flows in steady laminar motion in an annulus with an established velocity profile. For $x < 0$, both the fluid and the walls have a uniform temperature T_0. For $x > 0$, there is prescribed a surface temperature at the walls of the annulus. The problem is to find the temperature distribution and the variation of the heat transfer coefficient along the length of the annulus.

Subject to the limitations noted below, the energy equation describing the problem is

$$u\rho\, c_p \frac{\partial T}{\partial x} = \frac{k}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right) . \tag{1}$$

In writing Eq. (1) the following assumptions are made:

(a) The physical properties of the fluid are constant.

(b) The viscous energy dissipation is negligible.

(c) The axial diffusion of heat is negligible compared to the radial diffusion.

In the immediate region downstream from a step change in wall temperature, the axial temperature gradients could be large and of the same order of magnitude as the radial temperature gradients. The effect of axial temperature gradients on temperature distribution and heat transfer has been studied, for example, by Schneider(7) and Singh.(8) They found that the effect of axial heat conduction on heat transfer is negligible for $Pe \geq 10$.

The statement of the problem is completed by specifying the boundary conditions for the function $T(x,r)$. The following boundary conditions are considered:

$$\left.\begin{aligned}
&\text{Problem (a)}\\
&\qquad T(0,r) = T_0,\ r \neq r_i \neq r_0;\ \ T(x,r_i) = T(x,r_0) = T_w \text{ for } x \geq 0\\
&\text{Problem (b)}\\
&\qquad T(0,r) = T_0,\ r \neq r_i \neq r_0;\ \ T(x,r_i) = T_w,\ \left.\frac{\partial T}{\partial r}\right|_{r=r_0} = 0 \text{ for } x \geq 0\\
&\text{Problem (c)}\\
&\qquad T(0,r) = T_0,\ r \neq r_i \neq r_0;\ \ \left.\frac{\partial T}{\partial r}\right|_{r=r_i} = 0,\ T(x,r_0) = T_w \text{ for } x \geq 0
\end{aligned}\right\} \quad (2)$$

2.1.2 Fully Developed Velocity Profile in Laminar Flow

The velocity distribution for fully developed laminar flow in a concentric annulus with constant physical properties is given by Lamb.(10) Since the definition of the dimensionless radius used in this report is different from that of Lamb, the derivation of the velocity distribution is presented. The differential equation of motion for fully developed laminar flow is

$$\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right) = \frac{dp}{dx} \quad . \tag{3}$$

Introducing a dimensionless radius defined as $\xi = r/r_0$, Eq. (3) becomes

$$\frac{\mu}{\xi}\frac{\partial}{\partial \xi}\left(\xi\,\frac{\partial u}{\partial \xi}\right) = r_0^2\,\frac{dp}{dx} \quad . \tag{4}$$

The boundary conditions are:

$$u = 0 \text{ at } \xi = \xi_i (r = r_i)$$

and

$$u = 0 \text{ at } \xi = 1 (r = r_0) \quad . \tag{5}$$

Since for fully developed flow the pressure gradient is constant, the solution of Eq. (4) with the boundary conditions Eq. (5) becomes

$$u = -\frac{r_0^2}{4\mu}\left(\frac{dp}{dx}\right)\left[1 - \xi^2 - \frac{(1-\xi_i^2)}{\ell n\,\xi_i}\,\ell n\,\xi\right] \quad . \tag{6}$$

The average velocity, defined as

$$\overline{u} = \frac{\int_{\xi_i}^{1} u\xi \; d\xi}{\int_{\xi_i}^{1} \xi \; d\xi} \quad , \tag{7}$$

becomes

$$\overline{u} = -\frac{r_0^2}{4\mu}\left(\frac{dp}{dx}\right)\frac{\left[\frac{1}{4}(1-\xi_i^4) + \frac{(1-\xi_i^2)^2}{4\ell n\,\xi_i}\right]}{\frac{1}{2}(1-\xi_i^2)} = -\frac{r_0^2}{8\mu}\left(\frac{dp}{dx}\right)\left[1 + \xi_i^2 + \frac{(1-\xi_i^2)}{\ell n\,\xi_i}\right] \quad . \tag{8}$$

The ratio of the local velocity to the average velocity is given by:

$$\frac{u}{\overline{u}} = 2\,\frac{\left[1 - \xi^2 - \frac{(1-\xi_i^2)}{\ell n\,\xi_i}\,\ell n\,\xi\right]}{\left[1 + \xi_i^2 + \frac{(1-\xi_i^2)}{\ell n\,\xi_i}\right]} \quad . \tag{9}$$

It can be noted that the velocity profile is not symmetrical about the mid-point of the gap between the inside and outside radii. The point where maximum velocity occurs is shifted towards the inside wall of the annulus. When $\xi_i \rightarrow 1$, the flow in an annulus approaches the flow between two parallel plates.

2.1.3 Solution of the Problem

Introducing dimensionless variables, Eq. (1) and the boundary conditions Eq. (2) become

$$f\,\frac{\partial\theta}{\partial\zeta} = \frac{1}{\xi}\,\frac{\partial}{\partial\xi}\left(\xi\,\frac{\partial\theta}{\partial\xi}\right) \tag{10}$$

and

Problem (a)

$$\theta(0,\xi) = 1;\ \xi \neq \xi_i \neq 1;\ \theta(\zeta,\xi_i) = \theta(\zeta,1) = 0 \text{ for } \zeta \geq 0$$

Problem (b)

$$\theta(0,\xi) = 1;\ \xi \neq \xi_i \neq 1;\ \theta(\zeta,\xi_i) = 0,\ \left.\frac{\partial\theta}{\partial\xi}\right|_{\xi=1} = 0 \text{ for } \zeta \geq 0 \qquad (11)$$

Problem (c)

$$\theta(0,\xi) = 1;\ \xi \neq \xi_i \neq 1;\ \left.\frac{\partial\theta}{\partial\xi}\right|_{\xi=\xi_i} = 0,\ \theta(\zeta,1) = 0 \text{ for } \zeta \geq 0$$

respectively.

The method of separation of variables yields the solution

$$\theta = \sum_{n=0}^{\infty} c_n R_n(\xi) \exp(-\lambda_n^2 \zeta) \quad , \qquad (12)$$

in which $R_n(\xi)$ satisfies the equation

$$(\xi R_n')' + \lambda_n^2 f \xi R_n = 0 \qquad (13)$$

with the boundary conditions:

Problem (a)

$$R_n(\xi_i) = R_n(1) = 0$$

Problem (b)

$$R_n(\xi_i) = 0,\ R_n'(1) = 0 \qquad . \qquad (14)$$

Problem (c)

$$R_n'(\xi_i) = 0,\ R_n(1) = 0$$

Equation (13) with its boundary conditions Eq. (14) belongs to the well-known class of differential equations of the Sturm-Liouville type. Solutions are possible only for a discrete, though infinite, set of λ values. The set of constants c_n are now to be determined so that the condition

$$\theta(0,\xi) = 1,\ (\xi \neq \xi_i \neq 1) \quad ,$$

is satisfied. From the orthogonality property of the solutions, it can be shown[11] that the coefficients c_n are given by

$$c_n = \frac{\int_{\xi_i}^{1} \xi f R_n d\xi}{\int_{\xi_i}^{1} \xi f R_n^2 d\xi} \quad . \tag{15}$$

Integrating Eq. (13) with respect to ξ from ξ_i to 1, we obtain

$$\int_{\xi_i}^{1} \xi f R_n d\xi = -\frac{1}{\lambda_n^2}\left[R_n'(1) - \xi_i R_n'(\xi_i)\right] \quad . \tag{16}$$

It can be further shown (details omitted here) that for boundary conditions Eq. (11a)

$$\int_{\xi_i}^{1} \xi f R_n^2 d\xi = \frac{1}{2\lambda_n}\left\{\left[\frac{\partial R_n}{\partial \lambda_n}\;\frac{\partial R_n}{\partial \xi}\right]_{\xi=1} - \xi_i\left[\frac{\partial R_n}{\partial \lambda_n}\;\frac{\partial R_n}{\partial \xi}\right]_{\xi=\xi_i}\right\} \quad , \tag{17}$$

and the coefficients c_n can be expressed as

$$c_n = -\frac{2\left[\left(\frac{\partial R_n}{\partial \xi}\right)_{\xi=1} - \xi_i\left(\frac{\partial R_n}{\partial \xi}\right)_{\xi=\xi_i}\right]}{\lambda_n\left[\left(\frac{\partial R_n}{\partial \lambda_n}\;\frac{\partial R_n}{\partial \xi}\right)_{\xi=1} - \xi_i\left(\frac{\partial R_n}{\partial \lambda_n}\;\frac{\partial R_n}{\partial \xi}\right)_{\xi=\xi_i}\right]} \quad . \tag{18}$$

The λ_n^2, R_n, and c_n were found with the aid of an electronic analogue computer. The details of the numerical solution are given in Appendix A. It should be noted that the solution of Eq. (13) with the boundary conditions Eq. (14) can also be obtained by another method. For example, the solution of Eq. (13) with the boundary conditions Eq. (14c) can be expressed as

$$R(\xi) = \sum_{n=0}^{\infty} A_n\left[J_0(\beta_n\xi)Y_0(\beta_n\xi_i) - J_0(\beta_n\xi_i)Y_0(\beta_n\xi)\right] \quad , \tag{19}$$

where $\beta_n^2 = \lambda_n^2 f$. The eigenfunctions $R(\xi)$ are therefore an infinite series of Bessel functions of order zero. These vanish when $\xi = \xi_i$, and vanish also when $\xi = 1$, provided β_n is a root of the equation

$$J_0'(\beta_n)Y_0(\beta_n\xi_i) - J_0(\beta_n\xi_i)Y_0'(\beta_n) = 0 \quad . \tag{20}$$

However, the determination of the eigenvalues β_n and the coefficients A_n is very involved.[8] For this reason, the solution of Eqs. (13) and (14) was obtained on an analogue computer.

2.1.4 Expressions for Some Heat Transfer Parameters

From the temperature distribution, Eq. (12), various quantities of engineering interest can be determined. For example, the local heat flux, heat transfer coefficient, and Nusselt number may be determined from the definitions

$$q_i''(x) = -k \left.\frac{\partial T}{\partial r}\right|_{r=r_i} \quad ; \quad q_0''(x) = k \left.\frac{\partial T}{\partial r}\right|_{r=r_0} \quad ;$$

$$h = \frac{q''}{T_w - T_m} \quad ; \quad Nu = \frac{hD_e}{k} \quad , \tag{21}$$

where the mixing cup temperature is given by

$$T_m = \frac{\int_{r_i}^{r_0} ruT\, dr}{\int_{r_i}^{r_0} ru\, dr} \quad . \tag{22}$$

The local heat flux at the inner wall is given by

$$q_i''(\zeta) = -k \left.\frac{\partial T}{\partial r}\right|_{r=r_i} = \frac{k(T_w - T_0)}{r_0} \sum_{n=0}^{\infty} c_n R_n'(\xi_i) \exp(-\lambda_n^2 \zeta) \quad , \tag{23}$$

and at the outer wall by

$$q_0''(\zeta) = k \left.\frac{\partial T}{\partial r}\right|_{r=r_0} = -\frac{k(T_w - T_0)}{r_0} \sum_{n=0}^{\infty} c_n R_n'(1) \exp(-\lambda_n^2 \zeta) \quad . \tag{24}$$

The Nusselt number can be expressed as

$$Nu = \frac{q'' D_e}{(T_w - T_m)k} = \frac{q'' D_e}{\theta_m (T_w - T_0)} \quad , \tag{25}$$

where the dimensionless mixing cup temperature θ_m is defined analogously with Eq. (22):

$$\theta_m = \frac{\int_{\xi_i}^{1} \xi f \theta d\xi}{\int_{\xi_i}^{1} \xi f d\xi} = \frac{\sum_{n=0}^{\infty} c_n \exp(-\lambda_n^2 \zeta) \int_{\xi_i}^{1} \xi f R_n(\xi) d\xi}{\int_{\xi_i}^{1} \xi f d\xi} \quad . \tag{26}$$

It can be shown that

$$\int_{\xi_i}^{1} \xi f d\xi = \frac{1}{4} (1 - \xi_i^2) \quad . \tag{27}$$

Introducing Eqs. (16) and (27) in Eq. (26), we get

$$\theta_m = - \frac{4 \sum_{n=0}^{\infty} \frac{c_n}{\lambda_n^2} [R_n'(1) - \xi_i R_n'(\xi_i)] \exp(-\lambda_n^2 \zeta)}{(1 - \xi_i^2)} \quad . \tag{28}$$

Thus, the Nusselt numbers can be expressed as

$$Nu_i = - \frac{(1 - \xi_i)(1 - \xi_i^2) \sum_{n=0}^{\infty} c_n R_n'(\xi_i) \exp(-\lambda_n^2 \zeta)}{2 \sum_{n=0}^{\infty} \frac{1}{\lambda_n^2} c_n [R_n'(1) - \xi_i R_n'(\xi_i)] \exp(-\lambda_n^2 \zeta)} \quad , \tag{29}$$

and

$$Nu_0 = \frac{(1 - \xi_i)(1 - \xi_i^2) \sum_{n=0}^{\infty} c_n R_n'(1) \exp(-\lambda_n^2 \zeta)}{2 \sum_{n=0}^{\infty} \frac{1}{\lambda_n^2} c_n [R_n'(1) - \xi_i R_n'(\xi_i)] \exp(-\lambda_n^2 \zeta)} \tag{30}$$

at the inner and the outer surfaces, respectively.

The definition of heat transfer coefficient, and therefore of the Nusselt number, for example, as given by Eqs. (29) and (30) is not unique in the situation when heat is transferred from both surfaces. This has already been encountered by Jakob[9] and Seban.[12] This is because the mixing cup temperature, for a given velocity distribution [see Eq. (28)], depends not only on the heat flux at the surface in consideration, but also on the heat flux at the other surface.

When $\zeta \rightarrow 0$, $Nu \rightarrow \infty$. For values above a certain $\zeta = \zeta_e$, Nu will not differ by more than a few per cent from the final asymptotic value Nu_a. The region between 0 and ζ_e is called the thermal entrance region. In this region Nu decreases from an infinitely large value at $\zeta = 0$ to a value Nu_a for $\zeta > \zeta_e$. For large values of ζ only the first term of the series for Nu is of importance, so that

$$Nu_{a,i} = - \frac{(1 - \xi_i)(1 - \xi_i^2)\,\lambda_0^2 R_0'(\xi_i)}{2\,[R_0'(1) - \xi_i R_0'(\xi_i)]} \tag{31}$$

and

$$Nu_{a,o} = \frac{(1 - \xi_i)(1 - \xi_i^2)\,\lambda_0^2 R_0'(1)}{2\,[R_0'(1) - \xi_i R_0'(\xi_i)]} \tag{32}$$

are the asymptotic or fully developed Nusselt numbers at the inner and the outer walls, respectively.

Often in analysis an average heat flux and an average Nusselt number with respect to tube length is of more utility than the local heat flux and Nusselt number. If the average heat flux over the (reduced) length ζ is defined as

$$\overline{q}'' = \frac{1}{\zeta} \int_0^{\zeta} q''(\zeta)\,d\zeta \quad , \tag{33}$$

it can be shown that

$$\overline{q}_i'' = \frac{k(T_w - T_0)}{\lambda_n^2\,\zeta r_0} \sum_{n=0}^{\infty} c_n R_n'(1)\,[1 - \exp(-\lambda_n^2 \zeta)] \tag{34}$$

and

$$\overline{q}_0'' = - \frac{k(T_w - T_0)}{\lambda_n^2\,\zeta r_0} \sum_{n=0}^{\infty} c_n R_n'(1)\,[1 - \exp(-\lambda_n^2 \zeta)] \tag{35}$$

as average heat fluxes at the inner and outer surfaces, respectively.

If the average Nusselt number is defined as

$$\overline{Nu} = \frac{1}{\zeta} \int_0^{\zeta} Nu\,d\zeta \tag{36}$$

for the case of an insulated outside wall, it can be shown by substituting Eq. (29) into Eq. (36) that

$$\overline{Nu}_i = \frac{(1-\xi_i)(1-\xi_i^2)}{2\xi_i\zeta} \int_0^{\zeta} \frac{\sum_{n=0}^{\infty} c_n R_n'(\xi_i) \exp(-\lambda_n^2 \zeta)}{\sum_{n=0}^{\infty} \frac{c_n R_n'(\xi_i)}{\lambda_n^2} \exp(-\lambda_n^2 \zeta)} d\zeta \quad . \tag{37}$$

Recognizing the integrand as of the form $-d(\ell n\, y)$ and integrating, we find

$$\overline{Nu}_i = \frac{(1-\xi_i)(1-\xi_i^2)}{2\,\xi_i\zeta} \left[-\ell n \sum_{n=0}^{\infty} \frac{c_n R_n'(\xi_i)}{\lambda_n^2} \exp(-\lambda_n^2 \zeta) \right]_0^{\zeta} \quad . \tag{38}$$

Substituting the limits of integration and noting [from Eq. (28)] that

$$\sum_{n=0}^{\infty} \frac{c_n R_n'(\xi_i)}{\lambda_n^2} = \frac{(1-\xi_i^2)}{4\,\xi_i} \quad ,$$

we get

$$\overline{Nu}_i = \frac{(1-\xi_i)(1-\xi_i^2)}{2\,\xi_i\zeta} \ell n \left[\frac{(1-\xi_i^2)}{4\,\xi_i \sum_{n=0}^{\infty} \frac{c_n R_n'(\xi_i)}{\lambda_n^2} \exp(-\lambda_n^2 \zeta)} \right] \quad . \tag{39}$$

2.2 Discussion of Results

The first six eigenfunctions for several values of the ratio of the inner to the outer radius of the annulus are presented graphically in Figures B-a, B-b, and B-c for Problems (a), (b), and (c), respectively. The corresponding eigenvalues λ_n^2, coefficients c_n, and products $c_n R_n'(\xi_i)$ and $c_n R_n'(1)$ obtained in the investigation are given in Tables B-a, B-b, and B-c (see Appendix B) for Problems (a), (b), and (c), respectively.

To the author's knowledge, no analytical solutions have been obtained for the problems considered in this report, and therefore the accuracy of the results obtained on the analogue computer cannot be checked. Additional eigenfunctions are, however, needed to improve the accuracy of the results for $(1/Pe)(x/r_0) < 0.01$. Near the step change in the wall temperature ($x = 0$), the infinite series Eq. (12) converges slowly, and thus a large number of terms are needed. The evaluation of the higher modes of the eigenfunctions and eigenvalues of Eq. (13) becomes exceedingly more difficult, and the accuracy of the expansion coefficients decreases. Therefore, to obtain the solution of Eq. (11) as $x \rightarrow 0$ it would be advantageous to use the method of Leveque.[9]

It is not practicable to give temperature distributions as functions of radial and axial coordinates for all the problems solved. However, with the aid of eigenfunctions and quantities given in the Appendix B, it is now possible to calculate the temperature distributions and heat transfer parameters of interest. For practical purposes the mixing cup temperature, as defined by Eq. (26), is of greater interest than the transverse temperature distributions. Likewise, the values of heat transfer coefficient or Nusselt number and heat flux as a function of $(1/Pe)\ [(x/r_0)]$ are of practical importance.

In Figure 2 is a comparison of the longitudinal change of θ_m for various values of ξ_i for Problem (a). All curves have the vertical axis as a tangent at $\theta_m = 1$ and the horizontal axis as an asymptote which is approached practically exponentially from about $(1/Pe)[(x/r_0)] = 0.1$ onwards. As $(1/Pe)[(x/r_0)]$ increases, the temperature of the fluid approaches the surface temperature. From Figures 3 and 4, similar behavior can be noted for Problems (b) and (c). It is seen from the figures that values of θ_m for a given value of parameter ξ_i and $(1/Pe)[(x/r_0)]$ are smallest in Problem (a); then follows those of Problems (c) and (b). These trends in θ_m are expected and can readily be explained from the consideration of the energy balance on the coolant in the annulus.

Comparison of the ratio Nu/Nu_a obtained in this study for various values of the parameter ξ_i for the cases of insulated outside wall and for the insulated inside wall of the annulus is shown in Figures 5 and 6, respectively. The curves do not extend all the way to $x = 0$ because the series appearing in Eqs. (29) and (30) have been truncated by using only the first six terms. The boundary conditions for the problems considered require a uniform temperature distribution at the annulus entrance. This produces an infinite radial temperature gradient at the wall at $x = 0$, and thus $Nu \rightarrow \infty$ as $x \rightarrow 0$. It is seen from Figures 5 and 6 that, as x becomes very large, Nu becomes constant, corresponding to a constant coefficient of heat transfer.

The Nusselt numbers have not been calculated for the case when equal wall temperatures are prescribed at both the inside and the outside walls of the annulus since the Nusselt numbers are not uniquely defined in this case. Figures 7 and 8 show the variation of the local heat flux at the inside and outside wall, respectively.

The ratio of the Nusselt number obtained with one adiabatic wall to that of the Nusselt number with heat transfer at both walls of the annulus as considered here is shown in Figure 9 for $\xi_i = 0.5$. It is seen that the variation of this ratio with $x/(Per_0)$ is insignificant.

Let us now consider the asymptotic, or fully developed, Nusselt numbers. In Figure 10 the calculated values are plotted against ξ_i for both insulated inner and outer walls. Note that the case of heat transfer from the inside surface only when $\xi_i \rightarrow 0$, because of the finite amount of heat transferred by an infinitely small surface, must give rise to an infinitely great Nusselt number. Because the first eigenvalue at $\xi_i = 0.95$ could not be obtained very accurately on the analog computer, the Nusselt numbers from $\xi_i = 0.8$ to $\xi_i = 0.95$ are shown by dashed lines. As $\xi_i \rightarrow 1$, the annulus approaches a parallel-plate system, and the asymptotic Nusselt numbers for the heat transfer at the inside wall only approach those for the heat transfer at the outside wall only.

Of considerable practical importance is the knowledge of the conditions under which the entrance effects must be accounted for in heat transfer calculations. In particular, it is of interest to know the value of $(1/Pe)[(x/r_0)]_e$ for Problems (b) and (c). [For Problem (a) the entrance lengths obtained depend on the particular definition of the Nusselt number used and therefore have not been determined.] Therefore, Figure 11 was prepared so that the thermal entrance lengths can be calculated for given values of Pe and r_0. The thermal entrance length is defined here as that value of $(1/Pe)[(x/r_0)]$ at which the Nusselt number approaches to within 5% of its asymptotic (fully developed) value. Other authors have used a 1 or 2% criterion for this entry length, but experimental heat transfer data are rarely of sufficient accuracy to warrant use of the 1% definition for comparison.

The thermal entrance length decreases almost linearly with the parameter ξ_i. Note also that, as ξ_i gets nearer to unity, the thermal entrance lengths predicted for the heat transfer from the inside wall of the annulus only approach those for the heat transfer from the outside wall of the annulus only. This same conclusion can also be reached, as discussed previously, from physical arguments. It is expected that the thermal entrance lengths will be higher in problems with greater asymmetry of heat transfer at the walls of the annulus.

The thermal entrance lengths calculated by Murakawa[4] for water heated from the inside surface of the annulus are somewhat higher than the predictions of this study. However, the criterion on which the thermal entrance lengths were based is not stated, and the temperature of the coolant (or the Prandtl number) is not given in his paper.

3. HEAT TRANSFER IN AN ANNULUS WITH DIFFERENT BUT CONSTANT WALL TEMPERATURES AT THE INNER AND OUTER WALLS

3.1 Analysis

3.1.1 Introduction

The solutions for Problem (a), which are described in Section 2 of this report, apply when the two walls of the annulus are at the same constant temperature. In this section the problem is generalized to the situation in which the inner and outer walls of the annulus are at different but constant wall temperatures. The method used is that of superposition. The results obtained are general in that one wall of the annulus can be heated and the other can be cooled. The boundary conditions for Eq. (1) are

$$T(0,r) = T_0,\ r \neq r_i \neq r_0;\ T(x,r_i) = T_{w_i},\ T(x,r_0) = T_{w_0} \text{ for } x \geq 0 \quad . \tag{40}$$

The approach in solving the heat transfer problem for a fluid flowing in an annulus with asymmetric wall temperatures is similar to that of Seban,[12] Yih and Cermak,[13] and Schenk and Beckers.[6] However, since both fully developed and thermal entrance regions are studied, the splitting of the general problem into two simpler problems with different boundary conditions is similar to that of Ref. 13.

3.1.2 Method of Superposition

To solve the energy Eq. (1) with the boundary conditions (40) it is convenient to split the problem into two simpler ones. Since the energy equation is linear, the general solution can be obtained by superposition of the two simpler solutions.

Let U denote the general solution of Eq. (1) with the boundary conditions

$$U(0,r) = T_0,\ r \neq r_i \neq r_0;\ U(x,r_i) = T_{w_i},\ U(x,r_0) = T_0 \text{ for } x \geq 0 \quad , \tag{41}$$

and let V denote the general solution at Eq. (1) with the boundary conditions

$$V(0,r) = T_0,\ r \neq r_i = r_0;\ V(x,r_i) = T_0,\ V(x,r_0) = T_{w_0} \text{ for } x \geq 0 \quad . \tag{42}$$

Because of the linearity of Eq. (1), any sum of solutions is also a solution, and a proper addition of solutions U and V will yield a temperature distribution satisfying the boundary conditions of the general problem. Combining solutions U and V we get

$$T = U + V - T_0 \quad . \tag{43}$$

This equation can be written in the form

$$T = \left(\phi + \frac{\xi - \xi_i}{1 - \xi_i}\right)\left(T_0 - T_{w_i}\right) + T_{w_i} + \left(\psi + \frac{1 - \xi}{1 - \xi_i}\right)\left(T_0 - T_{w_0}\right) + T_{w_0} - T_0 \tag{44}$$

where

$$\phi(\zeta, \xi) = \frac{U - T_{w_i}}{T_0 - T_{w_i}} - \frac{\xi - \xi_i}{1 - \xi_i} \tag{45}$$

and

$$\psi(\zeta, \xi) = \frac{V - T_{w_0}}{T_0 - T_{w_0}} - \frac{1 - \xi}{1 - \xi_i} \quad . \tag{46}$$

The functions ϕ and ψ were defined in this manner so that the eigenfunctions and eigenvalues obtained in Section 2, Problem (a), could be utilized for the present problem, as will become apparent later.

The solution ϕ satisfies the energy equation

$$f \frac{\partial \phi}{\partial \zeta} = \frac{1}{\xi} \frac{\partial}{\partial \xi}\left(\xi \frac{\partial \phi}{\partial \xi}\right) + \frac{1}{\xi(1 - \xi_i)} \tag{47}$$

with the boundary conditions

$$\phi(0, \xi) = \frac{1 - \xi}{1 - \xi_i}; \ \phi(\zeta, \xi_i) = \phi(\zeta, 1) = 0 \quad . \tag{48}$$

Similarly, the solution ψ satisfies the energy equation

$$f \frac{\partial \psi}{\partial \zeta} = \frac{1}{\xi} \frac{\partial}{\partial \xi}\left(\xi \frac{\partial \psi}{\partial \xi}\right) - \frac{1}{\xi(1 - \xi_i)} \tag{49}$$

with the boundary conditions

$$\psi(0, \xi) = \frac{\xi - \xi_i}{1 - \xi_i}, \ \psi(\zeta, \xi_i) = \psi(\zeta, 1) = 0 \quad . \tag{50}$$

The temperature distribution given by Eq. (44) satisfies the energy equation because ϕ and ψ satisfy that equation. The agreement with the boundary conditions of the general problem is demonstrated below:

At $\zeta = 0,\ \xi \neq \xi_i \neq 1,\ T = T_0$

$$T = \left(\frac{1 - \xi}{1 - \xi_i} + \frac{\xi - \xi_i}{1 - \xi_i}\right)\left(T_0 - T_{w_i}\right) + T_{w_i} + \left(\frac{\xi - \xi_i}{1 - \xi_i} + \frac{1 - \xi}{1 - \xi_i}\right)\left(T_0 - T_{w_i}\right)$$

$$+ T_{w_0} - T_0$$

$$T = \left(T_0 - T_{w_i}\right) + T_{w_i} + \left(T_0 - T_{w_0}\right) + T_{w_0} - T_0$$

$$T = T_0$$

At $\zeta \geq 0,\ \xi = \xi_i,\ T = T_{w_i}$

$$T = (0 + 0)\left(T_0 - T_{w_i}\right) + T_{w_i} + (0 + 1)\left(T_0 - T_{w_0}\right) + T_{w_0} - T_0$$

$$T = T_{w_i}$$

At $\zeta \geq 0,\ \xi = 1,\ T = T_{w_0}$

$$T = (0 + 1)\left(T_0 - T_{w_i}\right) + T_{w_i} + (0 + 0)\left(T_0 - T_{w_0}\right) + T_{w_0} - T_0$$

$$T = T_{w_0}$$

We, thus, see that the boundary conditions are satisfied.

3.1.3 Solutions of the Problem

Although in Section 2 we obtained solutions for a homogeneous partial differential equation, Eqs. (47) and (49) are nonhomogeneous. The method of solution of nonhomogeneous partial differential equations used here is the same as that suggested by Miller.[14]

To obtain the solution of Eq. (47) with the boundary conditions Eq. (48), we introduce a new function defined by

$$\phi(\zeta, \xi) = y(\zeta, \xi) + z(\xi) \quad , \tag{51}$$

where $z(\xi)$ is a function of ξ only and is to be determined. Therefore, we try to determine $z(\xi)$ so that $y(\zeta, \xi)$ satisfies the homogeneous equation. Substituting Eq. (51) into Eq. (47) we obtain

$$f\frac{\partial y}{\partial \zeta} = \frac{1}{\xi}\ \frac{\partial}{\partial \xi}\left(\xi\frac{\partial y}{\partial \xi}\right) + \frac{1}{\xi}\ \frac{\partial}{\partial \xi}\left(\xi\frac{\partial z}{\partial \xi}\right) + \frac{1}{\xi(1 - \xi_i)} \quad . \tag{52}$$

Hence, if $z(\xi)$ is such that

$$\frac{1}{\xi}(\xi z')' = -\frac{1}{\xi(1-\xi_i)} \quad , \tag{53}$$

$y(\zeta, \xi)$ will satisfy the homogeneous equation

$$f\frac{\partial y}{\partial \zeta} = \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial y}{\partial \xi}\right) \quad . \tag{54}$$

Equation (53) is readily integrated and gives

$$z(\xi) = -\frac{\xi}{1-\xi_i} + c_1 \ln \xi + c_2 \quad , \tag{55}$$

where c_1 and c_2 are arbitrary constants. Any value may be chosen for c_1 and c_2, and $y(\zeta, \xi)$ will still satisfy the homogeneous equation; however, c_1 and c_2 will be so determined that the boundary conditions assume the desirable form. From the second boundary condition of Eq. (48) and Eq. (51), we see that

$$\phi(\zeta, \xi_i) = y(\zeta, \xi_i) + z(\xi_i) = 0$$

or

$$y(\zeta, \xi_i) = -z(\xi_i) \quad .$$

Similarly, from the third boundary condition, we obtain

$$\phi(\zeta, 1) = y(\zeta, 1) + z(1) = 0 \quad ,$$

or

$$y(\zeta, 1) = -z(1) \quad .$$

The constants c_1 and c_2 will be so chosen that

$$z(\xi_i) = z(1) = 0 \quad .$$

Thus, from Eq. (55) we obtain

$$z(1) = 0 = \frac{1}{1-\xi_i} + c_2$$

or

$$c_2 = \frac{1}{1-\xi_i} \quad ,$$

and

$$z(\xi_i) = 0 = \frac{\xi_i}{1 - \xi_i} + c_1 \ln \xi_i + \frac{1}{1 - \xi_i}$$

or

$$c_1 = - \frac{1}{\ln \xi_i} \quad .$$

The solution for $z(\xi)$ then becomes

$$z(\xi) = \frac{1 - \xi}{1 - \xi_i} - \frac{\ln \xi}{\ln \xi_i}$$

With these values of c_1 and c_2, the function $y(\zeta, \xi)$ satisfies the homogeneous equation

$$f \frac{\partial y}{\partial \zeta} = \frac{1}{\xi} \frac{\partial}{\partial \xi} \left(\xi \frac{\partial y}{\partial \xi} \right) \tag{57}$$

and the boundary conditions

$$y(0, \xi) = \phi(0, \xi) - z(\xi) = \frac{\ln \xi}{\ln \xi_i}$$

$$y(\zeta, \xi_i) = y(\zeta, 1) = 0 \quad . \tag{58}$$

The method of solution of Eqs. (57) and (58) is identical with that given in Section 2.1.3 and, therefore, will not be repeated here. Note that the last two boundary conditions given by Eq. (58) are identical with those of Eq. (11a). Therefore, the eigenfunctions and eigenvalues for the present problem will be identical with those already found in Section 2 for Problem (a). The solution of Eq. (57) with boundary conditions Eq. (58) is

$$y = \sum_{n=0}^{\infty} C_n R(\xi) \exp(-\lambda_n^2 \zeta) \quad . \tag{59}$$

From the orthogonality property of the solutions, the coefficients are given by

$$C_n = \frac{\int_{\xi_i}^{1} \left(\frac{\ln \xi}{\ln \xi_i} \right) \xi f R_n \, d\xi}{\int_{\xi_i}^{1} \xi f R_n^2 \, d\xi} \quad . \tag{60}$$

To solve Eqs. (49) and (50), we introduce a new function defined as

$$\psi(\zeta, \xi) = Y(\zeta, \xi) + Z(\xi) \quad . \tag{61}$$

Using a procedure similar to that already discussed, we find that

$$Z(\xi) = -\frac{1-\xi}{1-\xi_i} + \frac{\ln \xi}{\ln \xi_i} \quad . \tag{62}$$

The function $Y(\zeta, \xi)$ satisfies the equation

$$f\frac{\partial Y}{\partial \zeta} = \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial Y}{\partial \xi}\right) \tag{63}$$

with the boundary conditions

$$\begin{aligned} Y(0, \xi) &= \psi(0, \xi) - Z(\xi) = 1 - \frac{\ln \xi}{\ln \xi_i} \quad ; \\ Y(\zeta, \xi_i) &= Y(\zeta, 1) = 0 \quad . \end{aligned} \tag{64}$$

The solution of Eqs. (63) and (64) is

$$Y = \sum_{n=0}^{\infty} D_n R_n(\xi) \exp(-\lambda_n^2 \zeta) \quad , \tag{65}$$

where the coefficient D_n is given by the relation

$$D_n = \frac{\int_{\xi_i}^{1} \left[1 - \left(\frac{\ln \xi}{\ln \xi_i}\right)\right] \xi f R_n \, d\xi}{\int_{\xi_i}^{1} \xi f R_n^2 \, d\xi} \quad . \tag{66}$$

Substituting Eqs. (56) and (59) into Eq. (51), we get

$$\phi(\zeta, \xi) = \sum_{n=0}^{\infty} C_n R_n(\xi) \exp(-\lambda_n^2 \zeta) + \frac{1-\xi}{1-\xi_i} - \frac{\ln \xi}{\ln \xi_i} \quad , \tag{67}$$

and inserting Eqs. (62) and (65) into Eq. (61) we obtain

$$\psi(\zeta, \xi) = \sum_{n=0}^{\infty} D_n R_n(\xi) \exp(-\lambda_n^2 \zeta) - \frac{1-\xi}{1-\xi_i} + \frac{\ln \xi}{\ln \xi_i} \quad . \tag{68}$$

Substituting Eqs. (67) and (68) into Eq. (44) we find the temperature distribution in an annulus with an unsymmetrically prescribed wall temperature:

$$T = \left(\sum_{n=0}^{\infty} C_n R_n(\xi) \exp(-\lambda_n^2 \zeta) + 1 - \frac{\ell n\, \xi}{\ell n\, \xi_i} \right) \left(T_0 - T_{w_i} \right) + T_{w_i}$$

$$+ \left(\sum_{n=0}^{\infty} D_n R_n(\xi) \exp(-\lambda_n^2 \zeta) + \frac{\ell n\, \xi}{\ell n\, \xi_i} \right) \left(T_0 - T_{w_0} \right) + T_{w_0} - T_0 \quad . \tag{69}$$

Note that $D_n = c_n - C_n$, and, in the special case that $T_{w_i} = T_{w_0} = T_w$, we get

$$T = \left[\sum_{n=0}^{\infty} C_n R_n(\xi) \exp(-\lambda_n^2 \zeta) + 1 - \frac{\ell n\, \xi}{\ell n\, \xi_i} \right.$$

$$\left. + \sum_{n=0}^{\infty} (c_n - C_n) R(\xi) \exp(-\lambda_n^2 \zeta) + \frac{\ell n\, \xi}{\ell n\, \xi_i} \right] (T_0 - T_w) + 2\, T_w - T_0$$

or

$$T = (T_0 - T_w) \left[\sum_{n=0}^{\infty} c_n R_n(\xi) \exp(-\lambda_n^2 \zeta) \right] + T_w \quad . \tag{70}$$

This is identical with Eq. (12).

The expressions for some heat transfer parameters follow readily from the definitions given in Section 2.1.4 [Eqs. (21) and (22)] and the temperature given by Eq. (69) and are, therefore, not repeated here.

3.2 Discussion of Results

For practical purposes the mixing cup temperature is frequently of greater use than the transverse temperature distribution. In addition, the value of Nusselt number as a function of the parameter $\frac{1}{Pe}\left(\frac{x}{r_0}\right)$ is of practical interest. However, it is not practicable to present all results of interest in this report for the range of parameters investigated. In view of the fact that the Nusselt number is not unique when heat is added at both surfaces, only heat fluxes are calculated.

By way of illustration, heat fluxes were computed for several values of the temperature ratio $\chi = \left(T_{w_i} - T_0\right)/\left(T_{w_0} - T_0\right)$ and the case when the ratio of the inside to the outside radius of the annulus is 0.5. The heat flux distributions at the outside wall of the annulus as obtained by substituting Eq. (69) into Eq. (21), which defines the heat fluxes, are shown in Figure 12 for various values of the parameter χ.

The results could be more readily understood if we note that, for either heating or cooling at both surfaces, we have the condition that $\chi > 0$. When one surface is heated and the other is cooled, we have that $\chi < 0$. The special case $\chi = 1$ corresponds to the situation when the temperatures at the inner and outer walls are the same, and the case $\chi = 0$ is for the problem when the inner wall is kept at the temperature T_0, and finally when $\chi = \pm\infty$, the outside wall is kept at the temperature T_0.

Note that for certain negative values of χ the heat flux parameter changes sign. From Figure 12, we see that for $\chi = -4$, heat is added to the fluid up to $\frac{1}{Pe}\left(\frac{x}{r_0}\right) = 0.03$, and for larger values of the abscissa heat is extracted from the coolant. One may also note that the length required to approach fully developed conditions is greater for unsymmetrically than for symmetrically prescribed wall temperatures of the annulus.

4. HEAT TRANSFER IN AN ANNULUS WITH ARBITRARY AXIAL WALL TEMPERATURE VARIATIONS

4.1 Analysis

4.1.1 Introduction

In engineering practice problems are frequently encountered in which the heating surface temperature is not constant, yet it is still required to be able to calculate heat transfer rate. Because of the linearity of the energy equation (1), a sum of solutions is again a solution. The method of superposition of solutions provides a powerful analytical tool for this purpose. It is thus possible to construct a solution for any kind of arbitrary variation of wall temperature with length by merely breaking the wall temperature up into a number of constant-temperature steps and using the solutions obtained in previous sections as a solution for each step.

The preceding results may be extended to include the cases for which the temperature of the inner and/or the outer walls of the annulus are arbitrary functions of the axial distance, $T_W(x)$, for $x \geq x_1 \geq 0$, through the use of Duhamel's formulae. This is identical with the superposition techniques employed in Refs. 5 and 13.

4.1.2 Generalization of Results of Section 2 - Arbitrary Wall Temperature Distribution

If the wall temperature has the distribution as shown in Figure 13, the temperature distribution in the annulus can be obtained from the solutions presented in Section 2 by superposition. The wall temperature distribution can be written mathematically as

$$
\begin{aligned}
T &= T_0 \text{ for } x \leq 0 \\
T &= T_W \text{ for } 0 \leq x \leq x_1 \\
T &= T_W(x) \text{ for } x \geq x_1
\end{aligned}
\tag{71}
$$

with $T_W(0) = T_0$. The dimensionless temperature can be expressed as

$$
\frac{T - T_0}{T_W - T_0} = 1 - \frac{T - T_W}{T_0 - T_W} = 1 - \theta(\zeta, \xi) \quad , \tag{72}
$$

where $\theta(\zeta, \xi)$ is given by Eq. (12). The solution for the present problem is obtained by means of Duhamel's formula[15] as

$$
T - T_0 = [1 - \theta(\zeta, \xi)](T_W - T_0) + \int_{\eta=\zeta_1}^{\eta=\zeta} [1 - \theta(\zeta - \eta, \xi)] \frac{d\, T_W(\eta)}{d\eta}\, d\eta \quad . \tag{73}
$$

The first term on the right hand side of Eq. (73) represents the temperature distribution in the fluid due to the step increase in the wall temperature at $\zeta = 0$. If, however, T_w has discontinuities at η_j, the integral is represented[5] by the summation of the discontinuities

$$\sum_{j=1}^{j} [1 - \theta(\zeta - \eta_j, \xi)][T_w(\eta_j^+) - T_w(\eta_j^-)] \quad . \tag{74}$$

The combination of the Riemann integral of Eq. (73) and the summation given by Eq. (74) may be represented by what is known as Stieltjes integral.[16] The derivative $d\, T_w(\eta)/d\eta$ is presumably a known function of η, and θ is a known solution of the problem with a step jump in the wall temperature.

The evaluation of the heat transfer rate at ζ for either the inner or the outer wall of the annulus follows readily from the definitions given in Eq. (21). The heat transfer coefficient and the Nusselt number can also be calculated. From Eqs. (31), (73), and (74) the local heat flux at the inner wall is given by

$$q_i''(\zeta) = \frac{k}{r_0}\left\{\theta'(\zeta, \xi_i)(T_w - T_0) + \int_{\eta=\zeta_1}^{\eta=\zeta} \theta'(\zeta-\eta, \xi_i)\frac{d\, T_w(\eta)}{d\eta}\, d\eta \right.$$
$$\left. + \sum_{j=1}^{j} \theta'(\zeta - \eta, \xi_i)[T_w(\eta_j^+) - T_w(\eta_j^-)]\right\} \quad . \tag{75}$$

Differentiation of Eqs. (73) and (74) with respect to ξ at $\xi = 1$ and substitution in Eq. (21) yield the heat flux at the outer wall:

$$q_0''(\zeta) = -\frac{k}{r_0}\left\{\theta'(\zeta, 1)(T_w - T_0) + \int_{\eta=\zeta_1}^{\eta=\zeta} \theta'(\zeta-\eta, 1)\frac{d\, T_w(\eta)}{d\eta}\, d\eta \right.$$
$$\left. + \sum_{j=1}^{j} \theta'(\zeta - \eta, 1)[T_w(\eta_j^+) - T_w(\eta_j^-)]\right\} \quad . \tag{76}$$

In general, for any given problem, the integral and summation of Eqs. (73) and (74) must be evaluated. In case $T_w(\eta)$ cannot be represented by a simple function, the integrals can be evaluated numerically.

4.1.3 Linear Wall Temperature Variation

For certain elementary types of wall-temperature variation, an analytical expression for $q''(\zeta)$ may be easily evaluated. As an example of an application of the method, the solutions will be obtained for Problems (b) and (c) for the wall-temperature variation illustrated in Figure 14. This includes a step in the wall temperature at $\eta = 0$ ($\zeta = 0$) and a linear variation of wall temperature thereafter. This type of wall temperature variation is of interest because it corresponds to the case giving rise to a fully established temperature profile (far away from the entrance) and a constant Nusselt number for the case of constant heat flux at the wall.

The wall temperature variation is expressed as

$$T_w = T_0 + a + b\eta \quad . \tag{77}$$

Substituting Eq. (77) into Eq. (76), including one step at $\eta = 0$, and substituting b for $dT_w/d\eta$, we obtain

$$q_0''(\zeta) = -\frac{k}{r_0}\left\{a \sum_{n=0}^{\infty} c_n R'(1) \exp(-\lambda_n^2 \zeta) + b \int_0^{\zeta} \sum_{n=0}^{\infty} c_n R'(1) \exp[-\lambda_n^2(\zeta - \eta)] d\eta \right\} \quad . \tag{78}$$

Performing the integration, substituting limits, and noting that

$$\sum_{n=0}^{\infty} c_n R_n'(1)/\lambda_n^2 = \frac{-(1 - \xi_i^2)}{4} \quad ,$$

Eq. (78) reduces to

$$q_0''(\zeta) = -\frac{k}{r_0}\left\{a \sum_{n=0}^{\infty} c_n R_n'(1) \exp(-\lambda_n^2 \zeta) - b\left[\frac{(1 - \xi_i^2)}{4} + \sum_{n=0}^{\infty} \frac{c_n R_n'(1)}{\lambda_n^2} \exp(-\lambda_n^2 \zeta)\right]\right\} \quad . \tag{79}$$

If desired, the local mixing cup temperature can be calculated by substituting Eq. (73) into Eq. (22). However, the mixing cup temperature can be obtained in a simpler fashion from the following consideration. Integrating Eq. (76) up to ζ to determine the total heat transfer rate up to this point, and then applying the energy balance to evaluate the mixing cup temperature at ζ, we get

$$q_0(\zeta) = 2\pi r_0 \frac{\mathrm{RePr}\ r_0}{1 - \xi_i} \int_0^{\zeta} q_0''(\zeta) d\zeta \tag{80}$$

and

$$q_0(\zeta) = \pi(r_0^2 - r_i^2)\,\bar{u}\rho c_p\,(T_m - T_0) \quad . \tag{81}$$

Combining these two equations, evaluating the integral, and substituting limits, we obtain for the mixing cup temperature

$$(T_m - T_0) = \frac{4}{(1-\xi_i^2)}\left\{ a\left[\sum_{n=0}^{\infty} \frac{c_n R_n'(1)}{\lambda_n^2} \exp(-\lambda_n^2\zeta) + \frac{(1-\xi_i^2)}{4}\right] + b\left[\frac{(1-\xi_i^2)}{4}\zeta - \sum_{n=0}^{\infty} \frac{c_n R'(1)}{\lambda_n^4}\exp(-\lambda_n^2\zeta) + \sum_{n=0}^{\infty}\frac{c_n R_n'(1)}{\lambda_n^4}\right]\right\} \tag{82}$$

or

$$T_m - T_0 = a + b\zeta + \frac{4}{(1-\xi_i^2)}\left\{ a \sum_{n=0}^{\infty} \frac{c_n R_n'(1)}{\lambda_n^2}\exp(-\lambda_n^2\zeta) + b\left[\sum_{n=0}^{\infty}\frac{c_n R_n'(1)}{\lambda_n^4} - \sum_{n=0}^{\infty}\frac{c_n R_n'(1)}{\lambda_n^4}\exp(-\lambda_n^2\zeta)\right]\right\} \quad . \tag{83}$$

If desired, the local heat transfer coefficient and Nusselt number can now be calculated. Substituting Eqs. (78) and (83) into Eq. (25) and noting that $T_W = T_0 + a + b\,\zeta$, we get

$$Nu_0 = \frac{(1-\xi_i)(1-\xi_i^2)}{2}\,\frac{\left\{ a\sum_{n=0}^{\infty} c_n R_n'(1)\exp(-\lambda_n^2\zeta) - b\left[\frac{(1-\xi_i^2)}{4} + \sum_{n=0}^{\infty}\frac{c_n R_n'(1)}{\lambda_n^2}\exp(-\lambda_n^2\zeta)\right]\right\}}{\left\{ a\sum_{n=0}^{\infty}\frac{c_n R_n'(1)}{\lambda_n^2}\exp(-\lambda_n^2\zeta) + b\left[\sum_{n=0}^{\infty}\frac{c_n R_n'(1)}{\lambda_n^4} - \sum_{n=0}^{\infty}\frac{c_n R_n'(1)}{\lambda_n^4}\exp(-\lambda_n^2\zeta)\right]\right\}} \tag{84}$$

Note that if b = 0, this equation reduces to Eq. (30), an equation for constant wall temperature. For large values of ζ all of the summations containing the exponentials go to zero and the asymptotic Nusselt number becomes

$$Nu_{a,0} = -\frac{(1-\xi_i)(1-\xi_i^2)^2}{8\sum_{n=0}^{\infty}\frac{c_n R_n'(1)}{\lambda_n^4}} \tag{85}$$

This series converges extremely rapidly. For example, for a value of parameter $\xi_i = 0.8$, the first term gives a value a fraction of a per cent smaller than the actual value.

Using a similar procedure, an equivalent expression is obtained for the local Nusselt number when the outer wall is insulated:

$$Nu_i = \frac{(1-\xi_i)(1-\xi_i^2)}{2\xi_i} \frac{\left\{ a \sum_{n=0}^{\infty} \frac{c_n R'_n(\xi_i)}{\lambda_n^2} \exp(-\lambda_n^2 \zeta) + b \left[\frac{(1-\xi_i^2)}{4\xi_i} - \sum_{n=0}^{\infty} \frac{c_n R'_n(\xi_i)}{\lambda_n^2} \exp(-\lambda_n^2 \zeta) \right] \right\}}{\left\{ a \sum_{n=0}^{\infty} \frac{c_n R'_n(\xi_i)}{\lambda_n^2} \exp(-\lambda_n^2 \zeta) + b \left[\sum_{n=0}^{\infty} \frac{c_n R'_n(\xi_i)}{\lambda_n^4} - \sum_{n=0}^{\infty} \frac{c_n R'_n(\xi_i)}{\lambda_n^4} \exp(-\lambda_n^2 \zeta) \right] \right\}} \tag{86}$$

and the asymptotic Nusselt number reduces to

$$Nu_{a,i} = \frac{(1-\xi_i)(1-\xi_i^2)^2}{8\xi_i^2 \sum_{n=0}^{\infty} \frac{c_n R'_n(\xi_i)}{\lambda_n^4}} \quad . \tag{87}$$

Heat fluxes, heat transfer coefficients, and Nusselt numbers can also be calculated for the case that heat is transferred at both surfaces of the annulus. Now, the temperature distributions obtained in Section 3 have to be used. Since in this problem the wall temperatures T_{w_i} and T_{w_0} have to be specified, no general results valid for all values of T_{w_i} and T_{w_0} can be calculated, and it is necessary to solve the problem for specific values of these parameters. For this reason, no calculations for the case are made. However, the heat transfer parameters of interest can be readily calculated from the results given in Tables B-a through B-d by means of a procedure identical with that employed in this section.

4.2 Discussion of Results

Figure 15 shows the variation of the asymptotic Nusselt numbers with parameter ξ_i for two cases: (1) insulated outside wall of the annulus; and (2) insulated inside wall of the annulus. Note that as $\xi_i \rightarrow 1$, the asymptotic Nusselt number for the situation when the outside surface is insulated approaches the Nusselt number for the case when the inside surface is insulated. This fact is also readily apparent from physical considerations. When $r_i \rightarrow r_0$ it does not make any difference which surface is heated and which is insulated, and the Nusselt numbers are therefore identical.

The results given in Figure 10 and Figure 15 are consistent in trend with those of Ref. 17 for laminar heat transfer in a pipe. Sellars et al. show that the asymptotic Nusselt numbers are higher when the wall temperature varies linearly with the axial distance than when it is constant. As was already mentioned, the case of linear variation of wall temperature corresponds to the case giving rise to fully established temperature profile and constant Nusselt numbers far away from the entrance for constant heat flux applied at the wall. Thus, the results given in Figure 15 are also valid for this latter boundary condition.

Appendix A

THE ANALOGUE COMPUTER SOLUTION OF EQUATION (13)

by

Louis C. Just

The analogue computer used in this study was an Electronics Association, Inc., Model 131 R. For specific information about the installation at Argonne, the reader is referred to Refs. 18 and 19.

Analogue computers of this type are limited to accuracies of 0.01%, an accuracy that can only be attained in the solution of simple linear equations. The complexity of this problem rules out accuracies this high. The formulation of $R_n''(\xi)$ used two dividers and two multipliers. These units have an accuracy within $\pm 0.025\%$ so that we can only be sure of $\pm 0.1\%$ in our answers. To further check the equipment and circuit, the problems were rechecked on another computer with a complete change in equipment.

$\ln\xi$ was generated directly by the machine, by means of the equation

$$\ln\xi = \int_{\xi_i}^{\xi} \frac{1}{\xi}\, d\xi + \ln\xi_i \quad .$$

Function generating equipment was considered, but this method was used because of its accuracy and convenience. Checks were performed and show that $\ln\xi$ was generated to within the accuracy of the equipment used (0.025% due to the formation of $1/\xi$).

The evaluation of the expansion coefficients c_n is subject to still more inaccuracy because of additional multiplications and divisions. Overall, c_n should be accurate within $\pm 0.15\%$.

The method of solving Eq. (13) consists of assuming a value of R_n or R_n' at $\xi = \xi_i$, whichever is not prescribed by the boundary conditions Eq. (14), and then integrating Eq. (13). The determination of the eigenvalues λ_n^2 was by a trial-and-error method of iteration. A value of λ_n^2 was assumed, and the integration of Eq. (13) was performed. If the appropriate boundary condition at $\xi = 1$ was not satisfied, an improved value of λ_n^2 was chosen. The process was repeated until the boundary condition at $\xi = 1$ was satisfied. This procedure has a source of error built in: it is up to the operator to decide how well the boundary condition is satisfied. A circuit for automatically holding the solution at $\xi = 1$ was used, thereby eliminating any inaccuracies due to reading R_n and R_n' at $\xi = 1 \pm \epsilon$.

In the course of this study, five values of ξ_i (0.05, 0.2, 0.5, 0.8, and 0.95) were investigated, with boundary conditions Eq. (14). The results for ξ_i = 0.95 are not given because the computer would not repeat the results. This nonrepeatability was probably due to the fact that high gains were involved.

It is believed that the results for boundary conditions Eq. (14b) are the least accurate. For these boundary conditions, $R_n'(1)$ was found to be "zero" for a range of eigenvalues λ_n^2. This insensitivity of λ_n^2 on $R_n'(1)$ can be partially eliminated by observing 100 $R_n'(\xi)$ instead of $R_n'(\xi)$. However, this does not completely eliminate all errors.

Appendix B

Table B-a

CONSTANTS FOR PROBLEM (a)

ξ_i	n	λ_n^2	c_n	$c_n R_n'(\xi_i)$	$c_n R_n'(1)$
0.05	0	13.17	-2.876	14.38	- 2.301
	1	69.33	1.120	-5.600	- 0.8960
	2	167.1	-1.609	8.047	- 1.481
	3	309.0	0.8709	-4.354	- 0.7838
	4	492.1	-1.319	6.594	- 0.9750
	5	718.9	0.7489	-3.744	- 0.7489
0.2	0	21.71	-1.557	7.887	- 3.076
	1	107.1	0.3917	-1.958	- 0.7736
	2	256.6	-0.9926	4.963	- 1.985
	3	470.6	0.3037	-1.518	- 0.6149
	4	750.2	-0.8202	4.108	- 1.702
	5	1095	0.2836	-1.418	- 0.6162
0.5	0	59.37	-1.712	8.585	- 5.881
	1	286.1	0.2054	-1.027	- 0.7036
	2	681.7	-1.111	5.855	- 3.833
	3	1246	0.1922	-0.9610	- 0.6727
	4	1983	-0.8980	4.490	- 3.143
	5	2889	0.1748	-0.8741	- 0.6424
0.8	0	370.9	-3.638	18.19	-15.93
	1	1757	0.2079	-1.039	- 0.9459
	2	4171	-2.386	11.93	-11.21
	3	7670	0.2253	-1.126	- 1.098
	4	12160	2.217	11.09	-11.25
	5	17590	0.1532	-0.7659	- 0.8234

Table B-b

CONSTANTS FOR PROBLEM (b)

ξ_i	n	λ_n^2	c_n	$c_n R_n'(\xi_i)$	$c_n R_n'(1)$
0.05	0	1.843	-1.801	9.005	0
	1	42.33	-0.5388	2.694	0
	2	124.7	-0.3084	1.542	0
	3	252.5	-0.2820	1.410	0
	4	419.3	-0.2444	1.222	0
	5	632.6	-0.2284	1.142	0
0.2	0	4.224	-0.9697	4.848	0
	1	66.57	-0.4315	2.158	0
	2	194.7	-0.3456	1.728	0
	3	387.6	-0.3037	1.518	0
	4	643.5	-0.2776	1.388	0
	5	966.1	-0.2529	1.265	0
0.5	0	15.29	-1.059	5.297	0
	1	182.8	-0.5978	2.989	0
	2	518.2	-0.4903	2.452	0
	3	1025	-0.4135	2.067	0
	4	1705	-0.3667	1.833	0
	5	2553	-0.3182	1.591	0
0.8	0	127.4	-2.992	11.50	0
	1	1120	-1.383	6.917	0
	2	3127	-1.170	5.850	0
	3	6144	-0.9885	4.942	0
	4	10180	-0.9778	4.489	0
	5	15230	-0.8226	4.113	0

Table B-c

CONSTANTS FOR PROBLEM (c)

ξ_i	n	λ_n^2	c_n	$c_n R_n'(\xi_i)$	$c_n R_n'(1)$
0.05	0	8.460	-1.426	0	- 1.783
	1	55.21	0.7181	0	- 1.269
	2	141.9	-0.4973	0	- 1.119
	3	270.5	0.3841	0	- 1.008
	4	441.1	-0.3175	0	- 0.9685
	5	661.4	0.2165	0	- 0.8823
0.2	0	10.89	-1.371	0	- 2.229
	1	77.97	0.6004	0	- 1.591
	2	208.1	-0.3858	0	- 1.379
	3	402.9	0.2843	0	- 1.237
	4	661.2	-0.2266	0	- 1.161
	5	984.2	0.1867	0	- 1.082
0.5	0	23.39	-1.312	0	- 3.869
	1	194.5	0.4806	0	- 2.739
	2	534.9	-0.2935	0	- 2.333
	3	1043	0.2132	0	- 2.117
	4	1723	-0.1634	0	- 1.944
	5	2573	0.1335	0	- 1.809
0.8	0	127.4	-1.266	0	-10.41
	1	1165	0.4203	0	- 7.292
	2	3231	-0.2509	0	- 6.154
	3	6630	0.1806	0	- 5.613
	4	10450	-0.1422	0	- 5.298
	5	15520	0.1163	0	- 5.001

Table B-d

EXPANSION COEFFICIENTS FOR GENERALIZED PROBLEM (a) - DIFFERENT TEMPERATURES AT THE INSIDE AND OUTSIDE WALLS OF THE ANNULUS

ξ_i	0.05		0.2		0.5		0.8	
n	C_n	D_n	C_n	D_n	C_n	D_n	C_n	D_n
0	-0.6924	-2.184	-0.5371	-1.020	-0.7307	-0.9813	-1.710	-1.928
1	-0.4291	1.549	-0.3685	0.7602	-0.5486	0.7540	-1.372	1.651
2	-0.3405	-1.268	-0.3216	-0.6710	-0.4823	-0.6287	-1.104	-1.282
3	-0.2969	1.168	-0.2889	0.5926	-0.4162	0.6084	-0.9854	1.211
4	-0.2490	-1.070	-0.2701	-0.5501	-0.4104	-0.4876	-0.8690	-1.348
5	-0.2273	0.9762	-0.2487	0.5323	-0.3394	0.5142	-0.7826	0.9358

NOMENCLATURE

Symbol	Definition
A_n	Coefficient in Equation (19)
a	Constant in Equation (77)
b	Constant in Equation (77)
C_n	Coefficient defined by Equation (60)
c_n	Coefficient defined by Equation (15)
c_p	Specific heat at constant pressure
c_1, c_2	Integration constants in Equation (55)
D_e	Equivalent diameter defined as $2(r_0 - r_i)$
D_n	Coefficient defined by Equation (66)
f	Function defined as $u/2\overline{u}$
h	Heat transfer coefficient
k	Thermal conductivity
Nu	Nusselt number defined by Equation (21)
Nu_{adb}	Nusselt number with one adiabatic wall of the annulus for Problem (a)
Pe	Peclet number defined as RePr
Pr	Prandtl number defined as $\mu c_p/k$
p	Pressure
Re	Reynolds number defined as $\rho \overline{u} D_e/\mu$
R_n	Eigenfunction obtained from the solution of Equation (13)
r	Radial coordinate
q	Heat transfer rate
q"	Heat flux
T	Temperature
T_m	Mixing cup temperature defined by Equation (22)
T_0	Temperature at the inlet to the annulus
T_w	Temperature at the wall
u	Local velocity
x	Axial coordinate
Y	Function defined by Equation (61)

Symbol	Definition
y	Function defined by Equation (51)
Z	Function defined by Equation (61)
z	Function defined by Equation (51)

Greek Symbols

β_n	Eigenvalue determined from Equation (20)
ζ	Dimensionless independent variable defined as $(1 - \xi_i)\, x/\mathrm{Pe}\, r_0$
η	Dummy independent variable
θ	Dimensionless temperature defined as $(T - T_w)/(T_0 - T_w)$
θ_m	Dimensionless mixing cup temperature defined by Equation (26)
λ_n^2	Eigenvalue satisfying Equation (13) and boundary conditions Equation (14)
μ	Dynamic viscosity
ξ	Dimensionless radial variable defined as r/r_0
ρ	Density
ϕ	Function defined by Equation (45)
χ	Temperature ratio defined as $(T_{w_i} - T_0)/(T_{w_0} - T_0)$
ψ	Function defined by Equation (46)

Subscripts

a	Designates the asymptotic value
e	Designates the entrance length
i	Designates a value of a variable of a function evaluated at the inside surface of the annulus
n	Designates the n^{th} eigenvalue or eigenfunction
o	Designates a value of a variable or a function evaluated at the outside surface of the annulus

Superscripts

'	Denotes differentiation with respect to ξ
—	Denotes an average value

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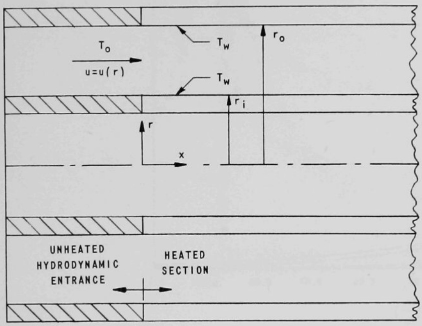

FIG. 1
DIAGRAM OF THE COORDINATE SYSTEM

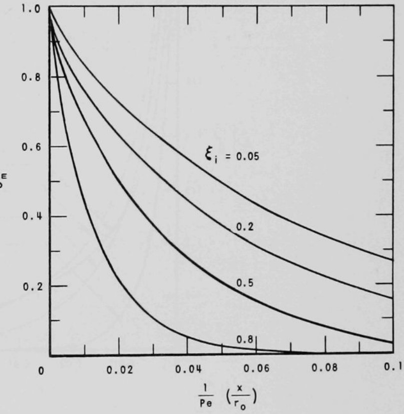

FIG. 2
VARIATION OF THE MIXING CUP TEMPERATURE WITH THE REDUCED DISTANCE FOR THE CASE OF EQUAL TEMPERATURES PRESCRIBED AT THE INSIDE AND OUTSIDE WALLS OF THE ANNULUS

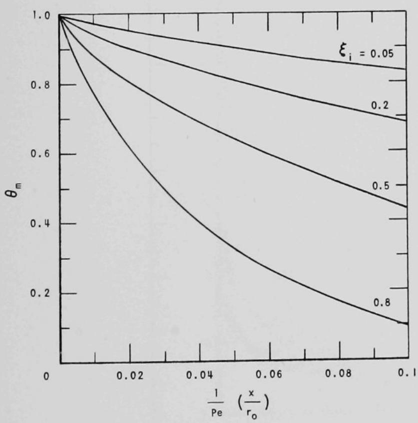

FIG. 3
VARIATION OF THE MIXING CUP TEMPERATURE WITH THE REDUCED DISTANCE FOR THE CASE OF INSULATED OUTSIDE WALL AND A TEMPERATURE STEP AT THE INSIDE WALL OF THE ANNULUS

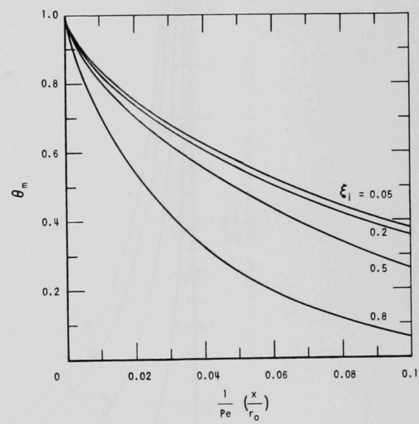

FIG. 4
VARIATION OF THE MIXING CUP TEMPERATURE WITH THE REDUCED DISTANCE FOR THE CASE OF INSULATED INSIDE WALL AND A TEMPERATURE STEP AT THE OUTSIDE WALL OF THE ANNULUS

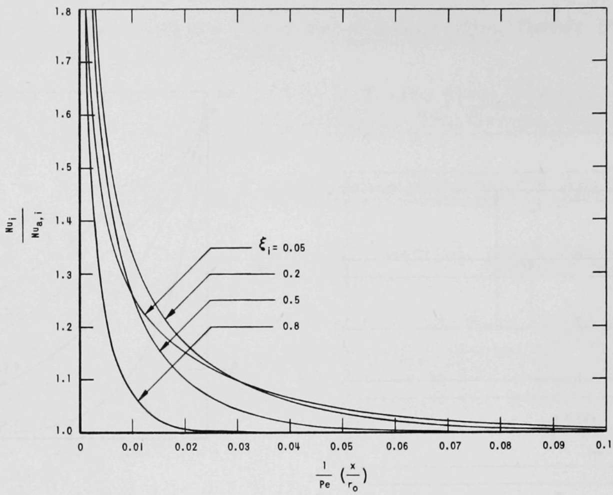

FIG. 5

VARIATION OF THE RATIO OF THE LOCAL NUSSELT NUMBER TO THE ASYMPTOTIC NUSSELT NUMBER FOR THE INSULATED OUTSIDE WALL

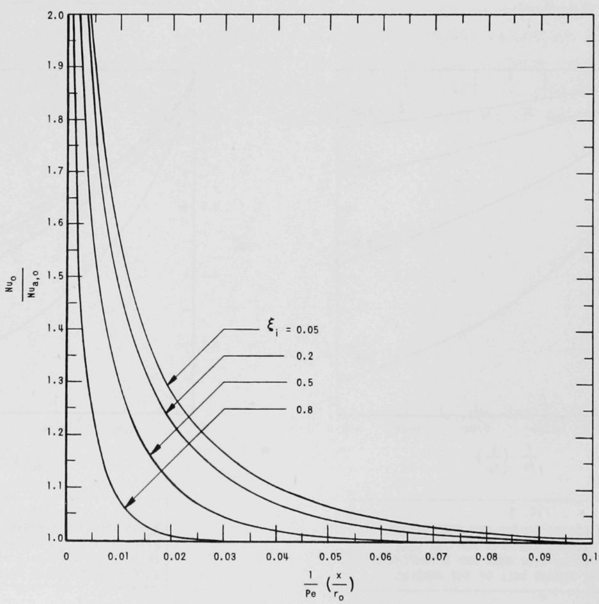

FIG. 6

VARIATION OF THE RATIO OF THE LOCAL NUSSELT NUMBER TO THE ASYMPTOTIC NUSSELT NUMBER FOR THE INSULATED INSIDE WALL

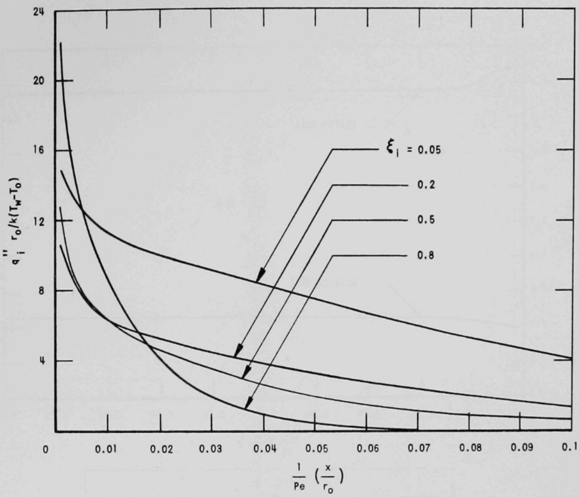

FIG. 7
VARIATION OF THE HEAT FLUX PARAMETER FOR THE INSIDE WALL WITH $(1/Pe)\ (x/r_o)$ AND ξ_i FOR EQUAL TEMPERATURES PRESCRIBED ON THE INSIDE AND OUTSIDE WALLS OF THE ANNULUS

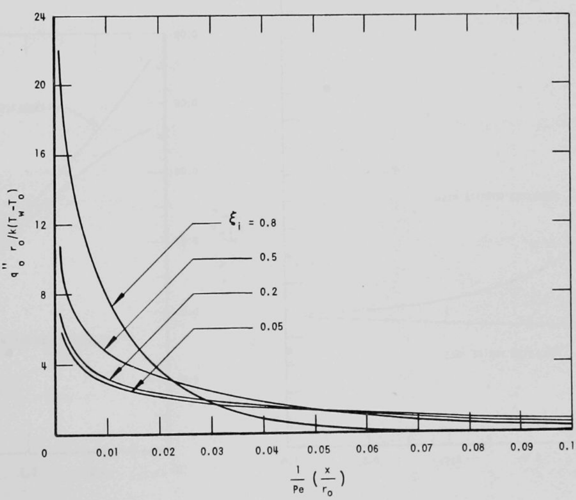

FIG. 8
VARIATION OF THE HEAT FLUX PARAMETER FOR THE OUTSIDE WALL WITH $(1/Pe)\ (x/r_o)$ AND ξ_i FOR EQUAL TEMPERATURES PRESCRIBED AT THE INSIDE AND OUTSIDE WALLS OF THE ANNULUS

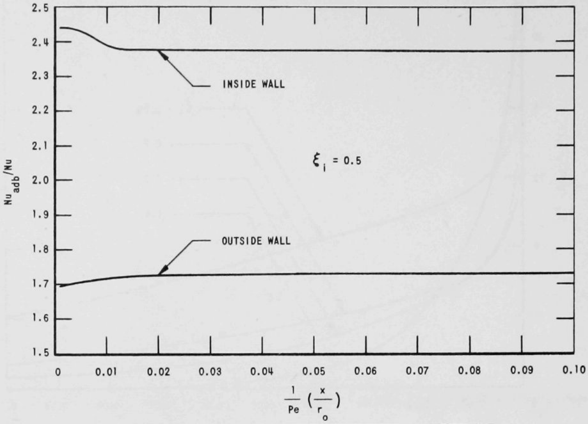

FIG. 9
VARIATION OF THE RATIO OF THE NUSSELT NUMBER WITH ONE ADIABATIC WALL TO THE NUSSELT NUMBER WITH HEAT TRANSFER AT BOTH WALLS OF THE ANNULUS

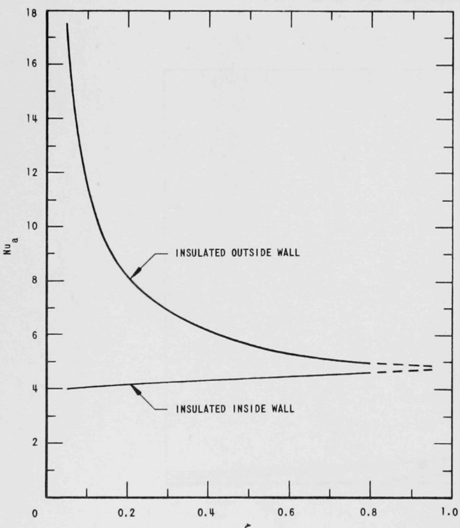

FIG. 10
DEPENDENCE OF THE ASYMPTOTIC NUSSELT NUMBER ON PARAMETER ξ_i FOR INSULATED INSIDE OR OUTSIDE WALLS OF THE ANNULUS

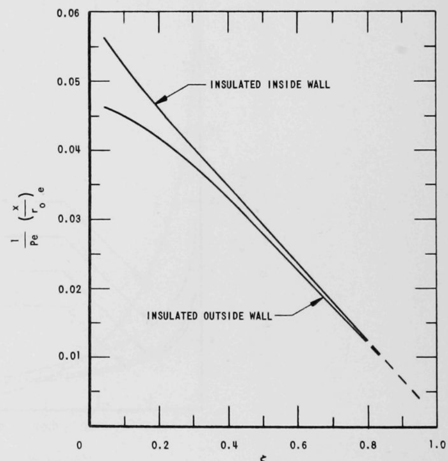

FIG. 11
VARIATION OF THE THERMAL ENTRANCE LENGTH WITH PARAMETER ξ_i

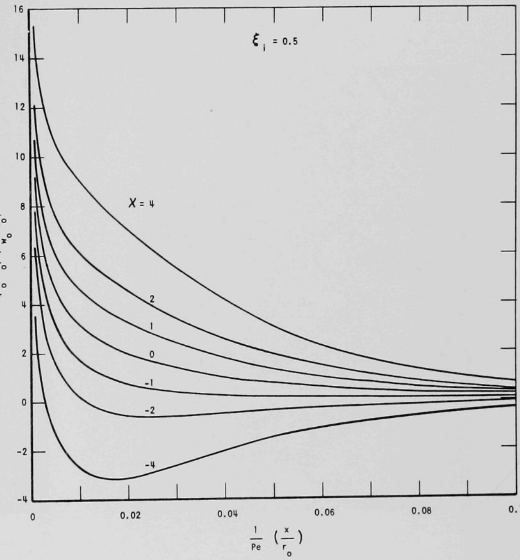

FIG. 12
VARIATION OF THE HEAT TRANSFER PARAMETER AT THE OUTSIDE WALL WITH REDUCED DISTANCE FOR UNSYMMETRICALLY PRESCRIBED TEMPERATURES AT THE WALLS OF THE ANNULUS

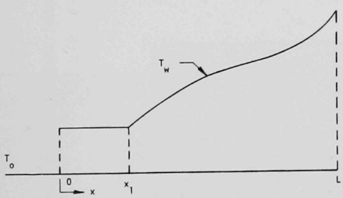

FIG. 13
ARBITRARY WALL TEMPERATURE VARIATION WITH THE AXIAL DISTANCE

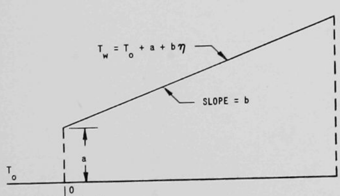

FIG. 14
LINEAR WALL TEMPERATURE VARIATION WITH THE AXIAL DISTANCE

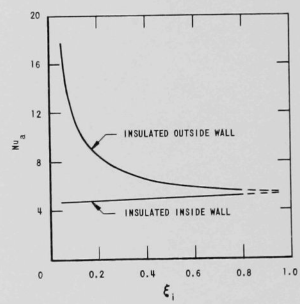

FIG. 15
DEPENDENCE OF THE ASYMPTOTIC NUSSELT NUMBER ON ξ_i FOR INSULATED INSIDE OR OUTSIDE WALLS OF THE ANNULUS WITH LINEAR WALL TEMPERATURE VARIATION

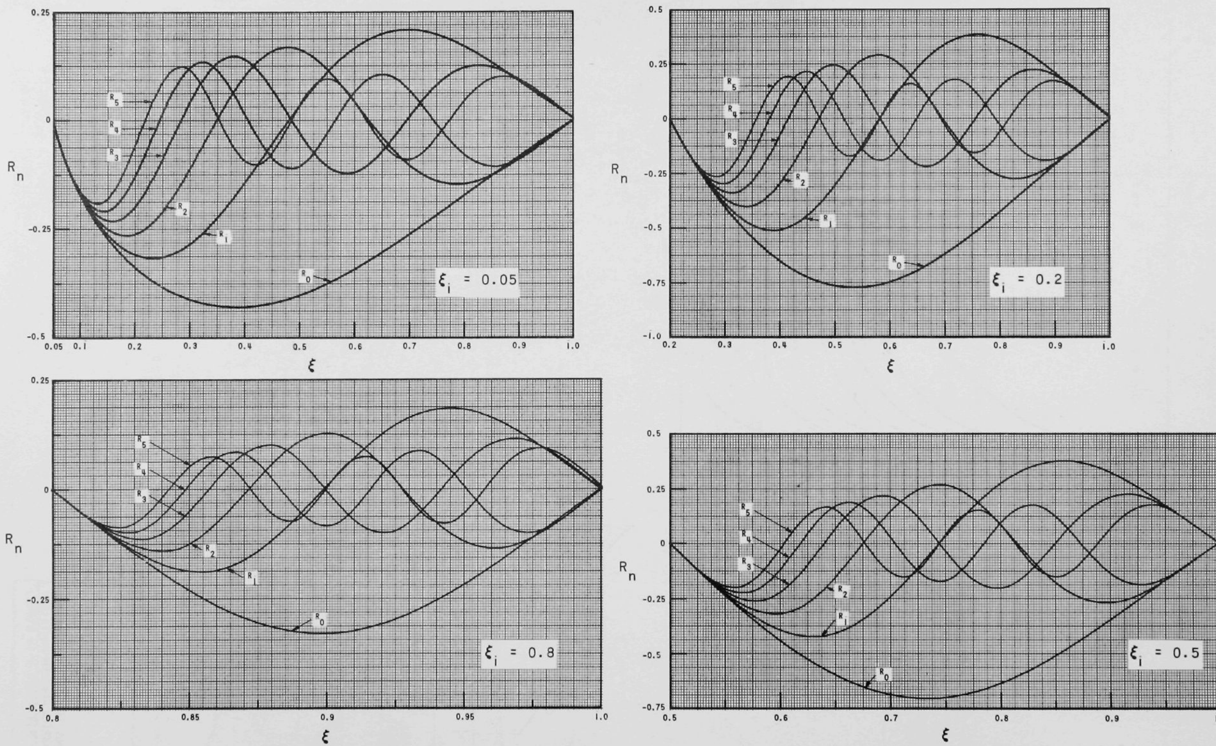

FIG. B-a

VARIATION OF THE FIRST SIX EIGENFUNCTIONS WITH ξ FOR BOUNDARY CONDITIONS (11a)

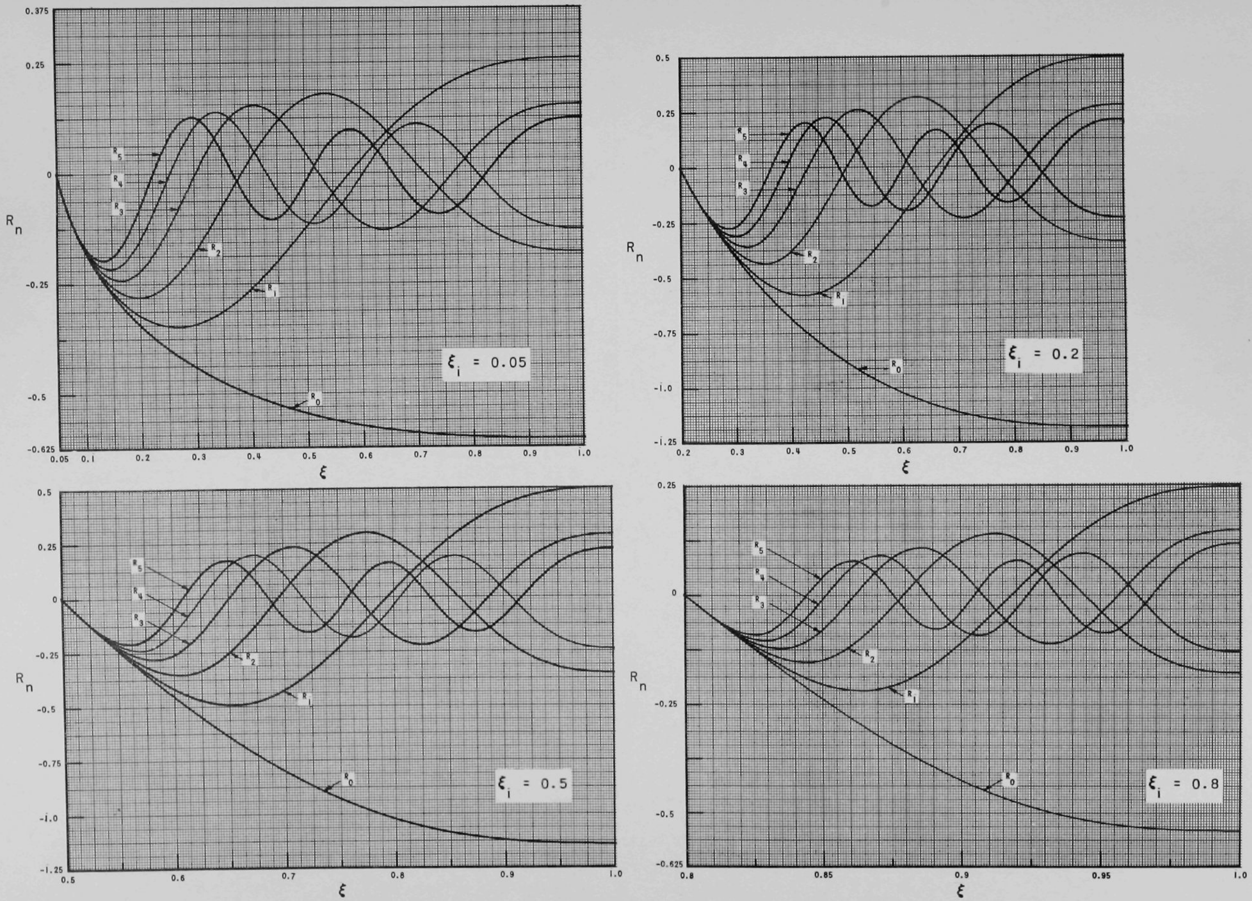

FIG. B-b

VARIATION OF THE FIRST SIX EIGENFUNCTIONS WITH ξ FOR BOUNDARY CONDITIONS (11b)

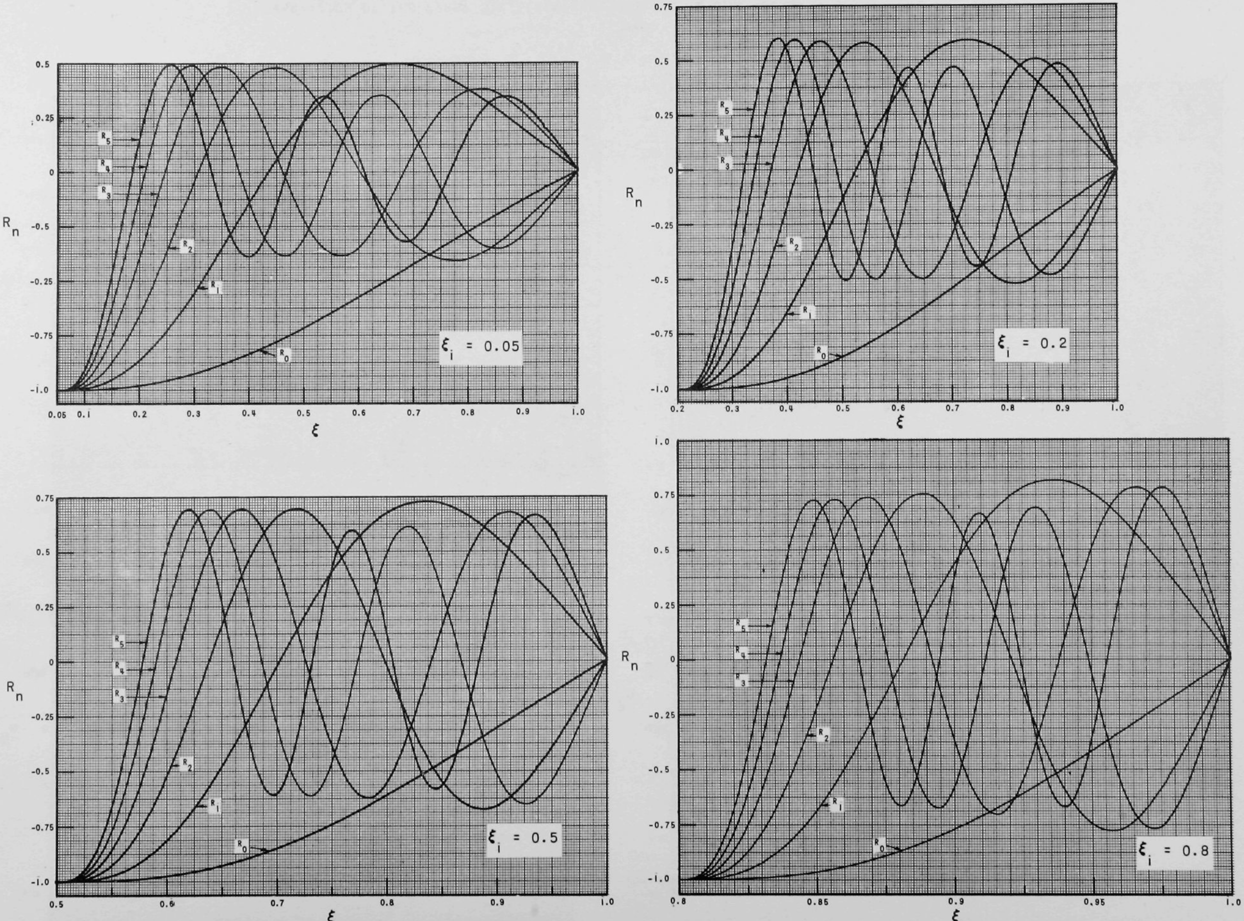

FIG. B-c
VARIATION OF THE FIRST SIX EIGENFUNCTIONS WITH ξ FOR BOUNDARY CONDITIONS (11c)

X